The Jordan-Brouwer Separation Theorem for Smooth Hypersurfaces

ELON L. LIMA
I.M.P.A., Estrada Dona Castorina 110, 22460 Rio de Janeiro, Brazil

We give here a simple proof of the following:

JORDAN-BROUWER SEPARATION THEOREM. *Let $M \subset \mathbb{R}^m$ be a connected, compact, orientable smooth hypersurface. Its complement $\mathbb{R}^m - M$ has two connected components, each of which has M as its point set boundary.*

A subset $M \subset \mathbb{R}^m$ is called a *smooth hypersurface* when every point $x \in M$ belongs to an open set U, on which is defined a smooth function $\varphi\colon U \to \mathbb{R}$ with the following properties: i) $\operatorname{grad}\varphi(x) \neq 0$; ii) $\varphi^{-1}(0) = M \cap U$. A vector $v \in \mathbb{R}^m$ is said to be *normal* to M at x when v is a multiple of $\operatorname{grad}\varphi(x)$. The *tangent space* to M at x is the set T_xM of all vectors in $\mathbb{R}^m$ that are perpendicular to $\operatorname{grad}\varphi(x)$. A map $f\colon M \to \mathbb{R}^n$ is called *smooth* when, for every $x \in M$, f is the restriction to $U \cap M$ of a smooth map $F\colon U \to \mathbb{R}^n$, defined on an open set U containing x. The *derivative* of a smooth map $f\colon M \to \mathbb{R}^n$ is the linear map $f'(x)\colon T_xM \to \mathbb{R}^n$, obtained by restriction of $F'(x)\colon \mathbb{R}^m \to \mathbb{R}^n$. (Recall that the matrix of $F'(x)$ is the Jacobian matrix of F.) A *diffeomorphism* is a smooth map with a smooth inverse. The Inverse Mapping Theorem says that if $f'(x)\colon T_xM \to \mathbb{R}^{m-1}$ is a linear isomorphism then f, restricted to some neighborhood V of x in M, gives a diffeomorphism of V onto an open subset of $\mathbb{R}^{m-1}$. (For more details, see Thorpe **[2]**.)

A smooth hypersurface $M \subset \mathbb{R}^m$ is said to be *orientable* when it admits a smooth field of normal unit vectors, i.e., when there exists a smooth map $v\colon M \to \mathbb{R}^m$ such that $|v(x)| = 1$ and $v(x)$ is normal to M at x, for every $x \in M$.

The assumption of orientability in Theorem A is redundant: any compact hypersurface must be orientable. (See Samelson **[1]** for a short proof of the smooth case.) Its presence, however, makes possible an easy proof. In many cases (for instance, when $M = S^{m-1}$) orientability is known *a priori*.

By "smooth" we mean C^∞. The proof holds verbatim for C^2 surfaces and, with a small technical modification (transverse, instead of normal fields) it would apply for C^1 surfaces as well.

Samelson's method of proof may also be used to get Theorem A, but we believe that our approach is more elementary. Instead of the Transversality Theorem and the classification of one-dimensional manifolds, we use the well-known fact that any smooth vector field $X\colon \mathbb{R}^m \to \mathbb{R}^m$, $X(x) = (a_1(x), \ldots, a_m(x))$, which fulfills the *integrability conditions* $\partial a_i/\partial x_j = \partial a_j/\partial x_i$ $(i, j = 1, \ldots, m)$, is the gradient of a smooth function $\varphi\colon \mathbb{R}^m \to \mathbb{R}$.

Once and for all, we fix a smooth field of unit normal vectors $v\colon M \to \mathbb{R}^m$ and define a smooth map $h\colon M \times \mathbb{R} \to \mathbb{R}^m$ by $h(x, t) = x + t \cdot v(x)$. For any given $\varepsilon > 0$, we denote by $h_\varepsilon\colon M \times (-\varepsilon, \varepsilon) \to \mathbb{R}^m$ the restriction of h.

The following standard result is included here for completeness.

LEMMA. *Let $M \subset \mathbb{R}^m$ be a compact, orientable, smooth hypersurface. For some $\varepsilon > 0$, h_ε: $M \times (-\varepsilon, \varepsilon) \to \mathbb{R}^m$ is a diffeomorphism onto an open subset of $\mathbb{R}^m$.*

Proof. For every $x \in M$, the derivative $h'(x,0)$: $T_xM \times \mathbb{R} \to \mathbb{R}^m$ is a linear isomorphism, since it sends each horizontal vector $(w,0)$ into w and each vertical vector $(0,t)$ into $t \cdot v(x)$ (which is perpendicular to w). By the Inverse Mapping Theorem, we may find $\delta_x > 0$ and an open neighborhood V_x of x in M such that h maps $V_x \times (-\delta_x, \delta_x)$ diffeomorphically onto an open neighborhood of x in $\mathbb{R}^m$. It remains only to show that, for some $\varepsilon > 0$, h_ε is injective. Assuming otherwise, we would find, for each $n \in \mathbb{N}$, *distinct* pairs $(x_n, s_n), (y_n, t_n)$ in $M \times (-1/n, 1/n)$ such that $h(x_n, s_n) = h(y_n, t_n)$. Since $M \times [-1,1]$ is compact we may assume (by taking subsequences, if necessary) that $x_n \to x \in M$, $y_n \to y \in M$, $s_n \to 0$ and $t_n \to 0$. Then

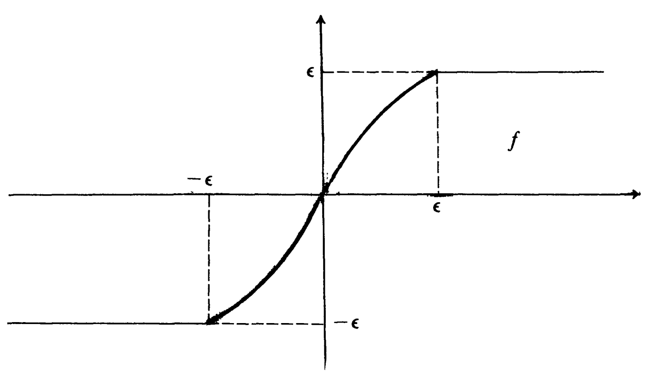
$$x = h(x,0) = \lim_n h(x_n, s_n) = \lim_n h(y_n, t_n) = h(y,0) = y.$$

Hence $\lim(x_n, s_n) = \lim(y_n, t_n) = (x,0)$. For all large values of n, (x_n, s_n) and (y_n, t_n) would belong to $V_x \times (-\delta_x, \delta_x)$ and then $h(x_n, s_n) \neq h(y_n, t_n)$. This contradiction proves the lemma.

We denote the image of h_ε by $V_\varepsilon(M)$ and call it a *tubular neighborhood of M*. We also write $V_\varepsilon[M] = h(M \times [-\varepsilon, \varepsilon])$.

Proof of the Theorem. We begin by showing that $\mathbb{R}^m - M$ is disconnected. More precisely, we define a smooth function φ: $\mathbb{R}^m \to \mathbb{R}$ such that $M = \varphi^{-1}(0)$ and $\operatorname{grad} \varphi(x) \neq 0$ for every $x \in M$. Then the open sets $A = \{x \in \mathbb{R}^m; \varphi(x) > 0\}$ and $B = \{x \in \mathbb{R}^m; \varphi(x) < 0\}$ are nonempty, disjoint, with $\mathbb{R}^m - M = A \cup B$. The definition of φ is as follows.

Let $V_{2\varepsilon}(M)$ be a tubular neighborhood of M. Take a smooth function f: $\mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$, $f'(t) > 0$ when $-\varepsilon < t < \varepsilon$, $f(t) = \varepsilon$ for $t \geqslant \varepsilon$ and $f(t) = -\varepsilon$ for $t \leqslant -\varepsilon$.



Any point in $V_{2\varepsilon}(M)$ may be written uniquely as $x + t \cdot v(x)$, with $x \in M$ and $|t| < 2\varepsilon$. Define a function g: $V_{2\varepsilon}(M) \to \mathbb{R}$ by $g(x + t \cdot v(x)) = f(t)$. Then g is smooth, $M = g^{-1}(0)$, $g(x + t \cdot v(x)) = \varepsilon$ for $t \in [\varepsilon, 2\varepsilon)$ and $g(x + t \cdot v(x)) = -\varepsilon$ for $t \in (-2\varepsilon, -\varepsilon]$. Let X: $\mathbb{R}^m \to \mathbb{R}^m$ be the vector field which equals $\operatorname{grad}\varphi g$ on $V_{2\varepsilon}(M)$ and vanishes outside this tubular neighborhood. The set $V_\varepsilon[M]$ is compact, hence closed in $\mathbb{R}^m$. X is smooth on each of the open sets $V_{2\varepsilon}(M)$ and $\mathbb{R}^m - V_\varepsilon[M]$ (in fact, identically zero on the latter). So, X is a smooth vector field in $\mathbb{R}^m$, which clearly fulfills the integrability conditions. Hence we may find a smooth function φ: $\mathbb{R}^m \to \mathbb{R}$ such that $\operatorname{grad}\varphi = X$.

By adding a constant to φ, if necessary, we may assume that $\varphi(x) = g(x)$ for every x in $V_{2\varepsilon}(M)$, since this is a connected open set on which φ and g have the same gradient. Moreover, φ is constant on every connected component of $\mathbb{R}^m - V_\varepsilon(M)$, since its gradient vanishes there. Now, every such component meets $V_{2\varepsilon}(M)$. [Given any $y \notin V_{2\varepsilon}(M)$, let p be a point in the closed set $V_\varepsilon[M]$ that minimizes distance from y. The line segment $[y, p]$ lies in $\mathbb{R}^m - V_\varepsilon(M)$ (hence in the component of y in this set) and meets $V_{2\varepsilon}(M)$ at all points near p.] Then $\varphi = \pm\varepsilon$ outside $V_\varepsilon(M)$ and $M = \varphi^{-1}(0)$. If $x \in M$ then $\operatorname{grad}\varphi(x) = c \cdot v(x)$, $c \in \mathbb{R}$, because the gradient is perpendicular to the level surface M. This gives

$$c = \langle \operatorname{grad}\varphi(x), v(x)\rangle = \frac{d}{dt}[g(x + t \cdot v(x))]_{t=0} = f'(0) = 0,$$

hence $\operatorname{grad} \cdot \varphi(x) \neq 0$ at each point $x \in M$.

Next, we show that the open sets A, B, defined at the beginning of the proof, are connected. In fact, A contains the connected set $P = h(M \times (0, 2\varepsilon)) = \{x + t \cdot v(x);\ x \in M,\ 0 < t < 2\varepsilon\}$. Moreover, every $y \in A$ is either in P or may be joined to a point $p \in P$ by a line segment $[y, p] \subset A$: just take p, (as before) as a point in the closed set $h(M \times [0, \varepsilon])$ that minimizes distance from y. A similar argument proves that B is connected.

Finally, we prove that M is the common point set boundary of both connected components A, B of $\mathbb{R}^m - M$.

For any $x \in M$, the point $x + t \cdot v(x)$ belongs to A when $0 < t < \varepsilon$ and to B when $-\varepsilon < t < 0$. Let $fr \cdot S$ denote the point set boundary of a set S. This shows that $x \in fr \cdot A \cap fr \cdot B$ for all $x \in M$. On the other hand, if $x \in fr \cdot A$ then $\varphi(x) \geqslant 0$ because $x \in \bar{A}$, but $\varphi(x) \leqslant 0$ because $x \notin A$, so $\varphi(x) = 0$. Therefore $fr \cdot A \subset M$, that is, $fr \cdot A = M$. Similarly, $fr \cdot B = M$ and the proof is finished.

REMARK. The same kind of argument applies when the hypersurface M, instead of compact, is assumed only to be a *closed* subset of $\mathbb{R}^m$. One needs only to change the Lemma, where instead of a constant $\varepsilon > 0$, a continuous positive function ε: $M \to \mathbb{R}$ must be found with the following property: if $x|y$ are in M then $x + s \cdot v(x)|y + t.v(y)$ for all $s \in (-\varepsilon(x), \varepsilon(x))$ and $t \in (-\varepsilon(y), \varepsilon(y))$. One may even go one step further and replace $\mathbb{R}^m$ by any simply-connected m-dimensional surface N containing the $(m-1)$-dimensional surface M as a closed subset. The same idea still applies, except that the construction of the tubular neighborhood $V_\varepsilon(M) \subset N$ is more subtle. (Instead of straight line segments one may use geodesics.)

REFERENCES

1. H. Samelson, Orientability of Hypersurfaces in $\mathbb{R}^n$, Proc. Am. Math. Soc., 22 (1969) pp. 301–302.
2. J. A. Thorpe, Elementary Topics in Differential Geometry, Springer Verlag, New York, 1979.
3. (Added in proof) B. Doubrovine, S. Novikov, A. Fomenko, Géométrie Contemporaine, Mir, Moscow, 1979. (Page 67, vol. 2 of this remarkable book contains a 10-line proof of the above result. That proof is wrong.)

The Königsberg Bridges—250 Years Later

PETER A. FOWLER
Department of Mathematics, California State University, Hayward, CA 94542

The year 1736 is generally accepted as marking the beginning of graph theory. In that year Euler published his article [2] on the Königsberg bridges problem and its generalizations, in which he demonstrated the necessity that every vertex have even degree. Remarkably, he dismissed the more difficult sufficiency argument as an "easy task... after a little thought." Not until 1873 was a sufficiency proof published [3]. Since that time there have been many proofs published. (See [1] for translations of [2] and [3] and for an interesting history of the problem.) The usual approach considers an Euler tour as a circuit to be traced out dynamically, ever enlarging until the graph is covered. In honor of the 250th anniversary of Euler's paper, we give a new proof in which we view the Euler tour as a static object in a graph and employ induction.

THEOREM. *A connected multigraph in which each vertex is of even degree has an Euler tour, that is, a circuit containing all the edges of the graph.*

Proof. We proceed by induction on the number of edges. If a graph has fewer than three vertices, the result is true by inspection, so consider a graph G with at least three vertices and n edges. Clearly, n is at least three.

Assume that an Euler tour exists in any connected graph with all even degree vertices and fewer than n edges. Select any vertex v of G. Since G is connected and all vertices have even degree, there exist two distinct edges which join v to (not necessarily distinct) vertices u and w. Delete these two edges. If $u = w$, insert a loop at u; otherwise, join u to w by an edge. The resulting graph has all vertices of even degree and has $n - 1$ edges. If it is connected, it follows from the inductive hypothesis that it has an Euler tour. An Euler tour for G can be obtained by replacing u, uw, w by u, uv, v, vw, w. If the new graph is disconnected, then it has exactly two components one of which contains v and the other of which contains u