Isometric immersions with semi-definite second quadratic forms

By

MANFREDO P. DO CARMO*) and ELON LIMA

Let $x\colon M \to \mathbf{R}^{n+N}$ be an isometric immersion of a compact, connected oriented n-dimensional riemannian manifold M in the euclidean space $\mathbf{R}^{n+N}$. We shall adopt the notations of CHERN-LASHOF [1], and assume some familiarity with this paper. B_ν is the bundle of unit normal vectors of $x(M)$, i.e., a point of B_ν is a pair $(p, \nu(p))$, where $\nu(p)$ is a unit normal vector to $x(M)$ at $x(p)$. If S_0^{n+N-1} is the unit sphere of $\mathbf{R}^{n+N}$, we define a map $\tilde{\nu}\colon B_\nu \to S_0^{n+N-1}$ by $\tilde{\nu}(p, \nu(p)) = \nu(p)$. If $d\Sigma$ is the volume element of S_0^{n+N-1} and dB is the volume element of B_ν, we define a function

$$G(p, \nu(p)) = G(p, \nu)$$

on B_ν by

$$\nu^* d\Sigma = G(p, \nu)\, dB\,.$$

Finally, the scalar product in $\mathbf{R}^{n+N}$ will be denoted by $\langle , \rangle$ and, for each point $(p, \nu(p)) \in B_\nu$, we define a quadratic form $\langle d^2x(p), \nu\rangle$ in the tangent space of M at p, called the second quadratic form of the immersion x in the normal direction ν.

We will prove the following

Theorem. *Assume all second quadratic forms of the immersion $x\colon M \to \mathbf{R}^{n+N}$ to be semi-definite, and definite at one point $(p, \nu_0(p)) \in B_\nu$. Then $x(M)$ belongs to a linear subvariety $\mathbf{R}^{n+1}$ of $\mathbf{R}^{n+N}$ and $x\colon M \to \mathbf{R}^{n+1}$ imbeds M as the boundary of a convex body; in particular M is homeomorphic to a sphere.*

Remark. Because the quadratic forms of the immersion x are assumed only to be semi-definite (and not definite), no numerical informations on the total curvature are immediately available from the hypothesis.

Lemma 1. *With the hypothesis of the theorem, let $\nu \in S_0^{n+N-1}$, $\nu = \tilde{\nu}(q, \nu(q))$, $q \in M$, be a regular value of $\tilde{\nu}$. Then the "height function" $h\colon M \to \mathbf{R}$ given by $h(p) = \langle x(p), \nu\rangle$, $p \in M$, has non-degenerate critical points, which are either maxima or minima.*

Proof. If $p \in M$ is a critical point of h, then

$$dh(p) = \langle dx(p), \nu\rangle = 0\,.$$

It follows that ν is a normal vector at $x(p)$ and the hessian

$$d^2h(p) = \langle d^2x(p), \nu\rangle$$

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is the second quadratic form at $(p, \nu(p))$. Because ν is a regular value of $\tilde{\nu}$, and $G(p, \nu)$ is, except for sign, the determinant of the above quadratic form, we conclude that p is a non-degenerate critical point. Moreover, since all eigenvalues of $d^2h(p)$ have the same sign, p is either a maximum or a minimum.

Lemma 2. *Let M be a compact manifold and $h: M \to \mathbf{R}$ a differentiable function on M with non-degenerate critical points which are either maxima or minima. Then h has exactly two critical points.*

Proof. By compactness, there is one critical point, say $q \in M$; let us assume for definiteness that q is a minimum. Let $\varphi(t)$ be a trajectory of grad h issuing from q, i.e., $\varphi(0)$ is close to q and $\lim\limits_{t \to -\infty} \varphi(t) = q$. By compactness, $\varphi(t)$ is defined for all $t > 0$.

Since h is bounded on M and

$$h(\varphi(t)) - h(\varphi(0)) = \int\limits_0^t \frac{d}{dt} h(\varphi(t))\, dt = \int\limits_0^t \| \operatorname{grad} h(\varphi(t)) \|^2,$$

we have that $\| \operatorname{grad} h \|$ is not bounded away from zero on $\varphi(t)$. Therefore, there is a critical point in the (compact) closure of the trajectory $\varphi(t)$. It follows that there exists the limit $\lim\limits_{t \to +\infty} \varphi(t) = p \in M$, and that p is a critical point of h. We shall say that $\varphi(t)$ goes into p.

Now, let S be a level surface of h sufficiently close to q. Let $A \subset S$ the set of points in S which are intersections of trajectories of grad h, issuing from q and going into p. By continuity and the fact that p and q are not saddles, A is an open set in S. On the other hand, a trajectory which issues from q and intersects S in a point belonging to the complement of A, goes into a critical point, say r. By the above argument, the complement of A is seen to be open in S. Because S is connected, $A = S$, and all the trajectories issuing from q go into p. By a similar argument, these trajectories cover an open and closed subset of M, hence the entire manifold M.

Therefore q and p are the only critical points of h in M, and the proof is finished.

Proof of the theorem. We first show that $x(M)$ is contained in a hyperplane of $\mathbf{R}^{n+N}$.

Assume that this is not true. By hypothesis, there exists a point $(p, \nu_0(p)) \in B_\nu$ such that $G(p, \nu_0) \neq 0$. Then by an argument of [1], lemma 1, it is possible to find a point $(p, \nu) \in B_\nu$ and points $x(q_1)$, $x(q_2)$, $q_1, q_2 \in M$, such that $G(p, \nu) \neq 0$ and $x(q_1)$, $x(q_2)$ lie on different sides of the tangent hyperplane at $x(p)$ normal to ν. Since $G(p, \nu) \neq 0$, the mapping $\tilde{\nu}$ is one-to-one in a neighborhood W of $(p, \nu) \in B_\nu$. Using SARD's theorem, we can find $(q', \nu') \in W$ such that ν' is a regular value of $\tilde{\nu}$ and $x(q_1)$, $x(q_2)$ still lie on different sides of the tangent hyperplane at $x(q')$ normal to ν'. It follows that the height function $\langle x, \nu' \rangle$ has at least three distinct critical points. By lemmas 1 and 2, this is a contradiction, and proves our assertion.

It is clear now that $x: M \to \mathbf{R}^{n+N-1}$ satisfies the hypothesis of the theorem, and, by induction, we conclude that $x(M)$ is contained in a linear subvariety of $\mathbf{R}^{n+1}$.

To prove that the immersion $x: M \to R^{n+1}$ imbeds M as the boundary of a convex body, consider the normal map $\nu: M \to S_0^n$. By lemmas 1 and 2, the inverse image of a regular element contains a unique element. Since M is compact and oriented, the degree of ν is ± 1. We now apply Theorem 4 of [1] to obtain the desired conclusion.

Remark. The above argument gives a new and simple proof of the fact that if $x\colon M \to \mathbf{R}^{n+1}$ is an isometric immersion of a compact riemannian manifold M with non-negative sectional curvatures, then $x(M) \subset \mathbf{R}^{n+1}$ is the boundary of a convex body and, in particular, M is homeomorphic to a sphere. This has been proved by CHERN-LASHOF [2], for $n = 2$. The general case follows from papers by HEIJENOORT [3] and SACKSTEDER [4] but no direct proof seems to be available.

References

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Anschrift der Autoren:

Manfredo P. do Carmo
Department of Mathematics
University of California
Berkeley, California 94720, USA

Elon Lima
Instituto de Matematica Pura e Aplicada
Rio de Janeiro, Brazil