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Commuting Vector Fields on S^3

By ELON L. LIMA*

In this paper an answer is given to the question, first posed to me by S. Smale, of whether the sphere S^3 admits a pair of commuting vector fields that are linearly independent at each point. The answer is no. The same problem was considered independently by J. Milnor who, in the list of problems of the Seattle Topology Conference (Summer 1963), defined the *rank* of a manifold M as the maximal number of commuting vector fields on M that are linearly independent at each point and asked, among other things, what is the rank of S^3.

In Milnor's terminology, our main result (stated and proved in § 4) is that every compact 3-manifold with finite fundamental group has rank one.

I am grateful to S. Smale and A. Haefliger for stimulating conversations on this subject, and to J. Stallings for useful information.

1. Preliminaries

We shall adopt the C^∞ point of view, so that the word *differentiable* is to mean *infinitely differentiable*. In this section we review the basic facts to be used later.

A *differentiable action* of a Lie group G on a differentiable manifold M is a differentiable map $\varphi: G \times M \to M$ such that $\varphi(gh, x) = \varphi(g, \varphi(h, x))$ and $\varphi(e, x) = x$ for any $g, h \in G$, $x \in M$, where e denotes the neutral element of G. For a given $g \in G$, we write $\varphi_g: M \to M$ to denote the map defined by $\varphi_g(x) = \varphi(g, x)$. Each φ_g is a diffeomorphism, whose inverse is $\varphi_{g^{-1}}$. The action φ induces, in a well-known way, a Lie algebra homomorphism $\varphi_*: \mathcal{G} \to \mathfrak{X}(M)$ of the Lie algebra of G into the space of all differentiable vector fields on M, called the *Lie homomorphism* of φ. Conversely, when M is compact and G is simply-connected, any Lie algebra homomorphism $h: \mathcal{G} \to \mathfrak{X}(M)$ is of the form $h = \varphi_*$ for some action, $\varphi: G \times M \to M$ [3, p. 82]. We shall use this fact mainly when $G = R^n$, and then, to give φ_* is the same as giving n differentiable vector fields $X_1, \cdots, X_n$ on M such that $[X_i, X_j] \equiv 0$, $i, j = 1, \cdots, n$. In this case, we say that the given vector fields *commute*. An action of the additive group of the reals is called a *flow*, so that a differentiable flow on a compact manifold M is equivalent to a differentiable vector field on M. Given 2 commuting vector fields X, Y on a compact manifold M, the relation between the flows

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$\xi, \eta: R \times M \rightarrow M$, generated respectively by X and Y, and the action $\varphi: R^2 \times M \rightarrow M$ determined by them is that, for any $r = (s, t) \in R^2$, $x \in M$, one has $\varphi_r(x) = \xi_s(\eta_t(x)) = \eta_t(\xi_s(x))$.

Given an action $\varphi: G \times M \rightarrow M$, the *orbit* of a point $x \in M$ is the set $\varphi(G \times x) = \{\varphi(g, x); g \in G\}$. The *isotropy group* of x is the set $G_x = \{g \in G; \varphi(g, x) = x\}$. G_x is a closed subgroup of G. Given $x \in M$, the map $g \rightarrow \varphi_g(x)$ of G onto the orbit of x induces, by passage to the quotient, a one to one continuous map $G/G_x \rightarrow M$ of the homogeneous space G/G_x onto the orbit of x. When φ is differentiable, such map is regular (i.e., has an injective derivative at each point) and, of course, when G/G_x is compact, it is a diffeomorphism of G/G_x onto the orbit of x.

We are interested in the cases $G = R$ and $G = R^2$. Given a differentiable flow $\xi: R \times M \rightarrow M$, the isotropy group of a point $x \in M$ may be $\{0\}$, a discrete subgroup with a generator $s_0 > 0$, or the whole real line. In the first case, the orbit of x is a non-compact one to one regular image of R. In the second case, the orbit of x is diffeomorphic to a circle and we say it is a *periodic orbit*, of period s_0, meaning that $\xi_s(x) = x$ if and only if $s = n \cdot s_0$, n integer. In the last case, x is a fixed point of the flow ξ, that is, $\xi_s(x) = x$ for every s. If X is the vector field associated to the flow ξ, we refer to the orbits of the flow ξ as *X-orbits* and a fixed point of ξ is a point $x \in M$ such that $X(x) = 0$, that is, a *singularity* of X.

When $G = R^2$, the isotropy group G_x of a point $x \in M$ under a differentiable action $\varphi: R^2 \times M \rightarrow M$ may be one of the following. It may be the whole R^2, and then x is a fixed point of φ: its orbit reduces to $\{x\}$. Next, suppose that $\dim G_x = 1$. Then G_x may either be a line through the origin of R^2, in which case $R^2/G_x \approx R$ and the orbit of x is a regular one to one image of R, or else G_x consists of a sequence of parallel lines $L + n \cdot v$, $n = 0, \pm 1, \pm 2, \cdots$ where L is a line through the origin of R^2 and $v \in R^2$ is a vector not contained in L. In this case $R^2/G_x \approx$ circle, so the orbit of x is a simple closed curve. Finally, the isotropy group G_x may have dimension 0, that is, it may be a discrete subgroup of the plane. This is the case we shall consider mostly. Here there are 3 possibilities. The first one is that $G_x = \{0\}$. Then the orbit of x is a regular one to one image of the plane R^2. The second is that $G_x = \{nv; n = 0, \pm 1, \pm 2, \cdots\}$ is a cyclic group with one non-zero generator $v \in R^2$. Then R^2/G_x is an infinite cylinder of which the orbit of x is a one to one continuous image. The last possibility is that G_x be a free abelian group on 2 generators: $G_x = \{mu + nv; m, n = 0, \pm 1, \pm 2, \cdots\}$ where $u = (a, b)$ and $v = (c, d)$ are linearly independent vectors in the plane R^2. Then R^2/G_x is a torus and the orbit of x is diffeomorphic to this torus. Again we remark that the orbit of x is compact if and only if

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R^2/G_x is compact. Let X, Y be commuting vector fields corresponding to the action $\varphi: R^2 \times M \to M$. A point $x \in M$ is fixed under φ if and only if $X(x) = Y(x) = 0$. The orbit of x under φ is 1-dimensional if and only if X and Y are linearly dependent (but not simultaneously zero) at x. The orbit of x is 2-dimensional if and only if $X(x)$ and $Y(x)$ are linearly independent.

Given the action $\varphi: G \times M \to M$, if two points $x, y \in M$ belong to the same orbit of φ, then the isotropy groups G_x and G_y are conjugate subgroups of G. So, when G is abelian, 2 points in the same orbit have the same isotropy groups. A set $X \subset M$ is said to be *invariant* under φ if, for any $x \in X$ and $g \in G$, $\varphi_g(x) \in X$. A subset $X \subset M$ is said to be a *minimal set* under φ if it is compact, invariant, non-empty, and contains no proper subset with these 3 properties. By Zorn's lemma, any non-empty invariant subset contains a minimal set. Given a minimal set $\mu \subset M$, the orbit of any point $x \in \mu$ is dense in μ, and from this it follows readily that any 2 points of μ have conjugate isotropy groups. When G is abelian, the isotropy group of all points in a given minimal set is the same.

The *ω-limit set* of a point $x \in M$ under a flow $\xi: R \times M \to M$ is the set $\xi_{+\infty}(x)$ of all points $y = \lim_n \xi_{s_n}(x)$, where $s_n \to \infty$. When M is compact, the ω-limit set $\xi_{+\infty}(x)$ of each point $x \in M$ is a non-empty, connected, invariant, compact subset of M.

To conclude this introductory section, we prove a topological lemma which is well known, but we need every part of its statement.

Let $V = V^{n-1}$ be a closed, connected $(n-1)$-manifold, topologically embedded in a compact, connected n-manifold $M = M^n$. The homology group $H_{n-1}(V; Z_2)$ has only one non-zero element. Denote by $i: V \subset M$ the inclusion map. When $i_*: H_{n-1}(V; Z_2) \to H_{n-1}(M; Z_2)$ is the zero homomorphism, we say that V *bounds* in M.

From now on we omit mention of the coefficient groups in homology, and cohomology, assuming them to be always Z_2.

LEMMA 1. *If V^{n-1} bounds in M^n, then*

(a) *$M - V$ has 2 connected components;*

(b) *V is the complete point-set boundary of each component of $M - V$;*

(c) *Given a coordinate system $x: \Omega \to R^n$ such that $x(\Omega)$ is a ball around the origin and $x(\Omega \cap V) = \{(x_1, \cdots, x_n) \in x(\Omega); x_n = 0\}$, let $a, b \in \Omega$ be points such that $x(a)$ and $x(b)$ lie in different sides of the hyperplane $x_n = 0$. Then a and b belong to distinct components of $M - V$.*

PROOF. We prove (c) first. Replacing, if necessary, Ω by a smaller domain, we may assume that x is also defined in the boundary of Ω. Clearly $\Omega - V$

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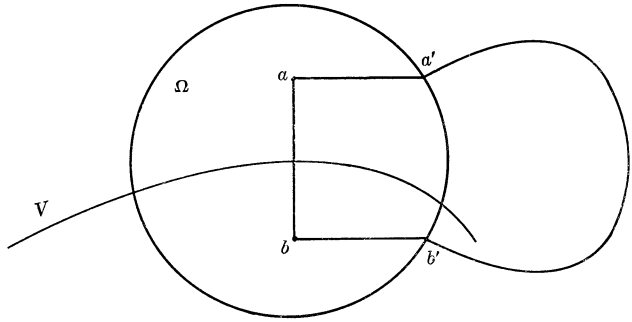

Fig. 1

has 2 connected components A, B, with $a \in A$, $b \in B$. Suppose, by contradiction, that a and b could be joined by an arc $\lambda_0 \subset M - V$. Of course, λ_0 is not contained in Ω. Let a' be the last point of λ_0 that lies in the boundary of A. Replace the part of λ_0 that goes from a to a' by an arc aa' such that $x(aa') =$ line segment joining $x(a)$ to $x(a')$. Do the same to b. Let λ_1 be an arc connecting a to b in Ω such that $x(\lambda_1) =$ line segment from $x(a)$ to $x(b)$. Then aa', the untouched middle part of λ_0, $b'b$ and λ_1 form a closed curve λ in M, intersecting V transversally and only at one point c. Hence the intersection number (mod 2) of V and λ is $V \sharp \lambda = 1$. On the other hand, V bounds in M, so $V \sharp \lambda = 0$. This contradiction proves (c).

The proofs of (a) and (b) may also be given by geometric arguments, provided that we assume V nicely embedded in M (as in the case we shall encounter). But it is neater—if less intuitive—to use the Lefschetz duality theorem, according to which $H_0(M - V) \approx H^n(M, V)$. In order to compute $H^n(M, V)$, we consider the cohomology exact sequence:

$$H^{n-1}(M) \to H^{n-1}(V) \to H^n(M, V) \to H^n(M) \to H^n(V) .$$

The first homomorphism is zero because it is the transpose of i_*. The last group is zero because dim $V = n - 1$. The second and fourth groups are isomorphic to Z_2. Hence $H^n(M, V) \approx Z_2 + Z_2$, so $H_0(M - V) \approx Z_2 + Z_2$, and $M - V$ has 2 connected components. This proves (a).

As to (b), it is clear that the point-set boundary of each component of $M - V$ is contained in V. To prove the converse inclusion, it suffices to show that, for any open ball $B \subset V$, the subset $V' = V - B$ does not disconnect M. Now V' is compact and $H^{n-1}(V') = 0$. Using duality and exact sequence as before, we see that $H_0(M - V') \approx H^n(M, V') \approx Z_2$, so $M - V'$ is connected.

2. The main argument

THEOREM 1. *Let X and Y be commuting vector fields on a compact*

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simply-connected 3-manifold M. Assume that X has a closed orbit and no singularities on M. Then the action of R^2 on M determined by X and Y has a compact orbit. (That is, a circle or a torus.)

PROOF. Denote by $\xi, \eta\colon R \times M \to M$ the flows generated by X and Y respectively, and let $\varphi\colon R^2 \times M \to M$ be the corresponding action of R^2 on M so that, for $r = (s, t) \in R^2$ and $x \in M$, $\varphi_r(x) = \xi_s\eta_t(x) = \eta_t\xi_s(x)$. Consider a point $x_0 \in M$ whose ξ-orbit is closed, say of period $s_0 > 0$, so $\xi_{s_0}(x_0) = x_0$, but $\xi_s(x_0) \neq x_0$ for $0 < s < s_0$. Call K the closure of the φ-orbit of x_0. By continuity, $\xi_{s_0}(x) = x$ for every $x \in K$. Since X has no singularities, the ξ-orbit of every $x \in K$ is then closed, with a period of the form s_0/n, $n > 0$ an integer. Clearly K is a compact, φ-invariant, non-empty subset of M, so we may choose a minimal set $\mu \subset K$ for the action φ.

The X-orbit of every point $x \in \mu$ is closed, with the same period $s_1 = s_0/n_1$. In fact, by the minimality of μ, given any two points $x, y \in \mu$, each of them belongs to the closure of the φ-orbit of the other. Suppose that, for some $s \in R$, $\xi_s(x) = x$. We may write $y = \lim_n \varphi_{r_n}(x)$, $r_n \to \infty$ in R^2. Hence

$$\xi_s(y) = \lim \xi_s\varphi_{r_n}(x) = \lim \varphi_{r_n}\xi_s(x) = \lim \varphi_{r_n}(x) = y\ .$$

Interchanging the roles of x and y, we see that $\xi_s(x) = x$ if, and only if, $\xi_s(y) = y$. Therefore, the X-orbits of x and y have the same period s_1.

Pick a point $x_1 \in \mu$. If $X(x_1)$ and $Y(x_1)$ are colinear, then the φ-orbit of x_1 is the same as its X-orbit, so it is a circle, and the theorem is proved. We may, therefore, assume that $X(x_1)$ and $Y(x_1)$ are linearly independent. Let γ_1 denote the (closed) orbit of x_1 under X. For each $x \in \gamma_1$, the vectors $X(x)$ and $Y(x)$ are linearly independent. The φ-orbit of x_1 is either a one to one regular image of a cylinder or it is a torus. We exclude the latter case, because it is what we want to prove.

Let x be an arbitrary point of γ_1. Given any neighborhood U of γ_1 in M, there are arbitrarily large positive values of t such that the point $\eta_t(x)$, in the Y-orbit of x, returns to U. This means that the ω-limit set $\eta_{+\infty}(x)$ of the Y-orbit of x intersects γ_1. In order to see this, let $L = \xi(R \times \eta_{+\infty}(x))$ be the union of all the X-orbits of points in $\eta_{+\infty}(x)$. The X-orbit of every point of $\eta_{+\infty}(x)$ has period s_1 because $\eta_{+\infty}(x) \subset \mu$. So $L = \xi\big([0, s_1] \times \eta_{+\infty}(x)\big)$, hence L is compact. Since L is evidently non-empty, invariant, and contained in μ, we must have $L = \mu$. In particular, $x \in L$. This means $X = \xi_s(y)$ for some $s \in R$ and some $y \in \eta_{+\infty}(x)$. Then $y = \xi_{-s}(x)$, so $y \in \gamma_1$ and consequently $y \in \gamma_1 \cap \eta_{+\infty}(x)$, as we wanted to show.

Let S be an *open* 2-dimensional cylindrical band, having γ_1 as its equator and transversal to all Y-orbits that intersect it. S will be constructed as a

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narrow ruled surface spanned by geodesic segments normal to the cylindrical φ-orbit along γ_1 (in some riemannian metric of M). Replacing, if necessary, S by a smaller band relatively compact in it, we may assume that there exists $\varepsilon > 0$ such that the segments of Y-orbit $I_x = \{\eta_t(x); |t| < \varepsilon\}$, of *length* 2ε and origin $x \in S$, form a (trivial) fibration, with base S, of the open set $V = \bigcup I_x$, $x \in S$. Denote by π: $V \rightarrow S$ the projection map that assigns to each $\eta_t(x) \in V$ its origin $x \in S$.

We choose S so narrow that the geodesic segments that generate it are transversal not only to the φ-orbit containing γ_1, but also to the φ-orbits of all points $x \in S$. Notice that, since the φ-orbit of every point of μ is dense in μ, there are points $x \in S \cap \mu$ arbitrarily close to γ_1, so close that the X-orbit of x lies entirely in V, hence it may be projected by π into S. By the previous remark, such projection is a simple closed curve in S, transversal to the generatrices of S, hence homotopic to γ_1 (and disjoint from it) in S. From this it follows that we may narrow S down further, in such a way that its absolute boundary $\bar{S} - S = \partial S$ consists of 2 of those curves, that is, two circles each of which lies entirely in one cyclindrical φ-orbit in μ, but not in the φ-orbit containing γ_1. Assume, from now on, that S has this property.

Define a real-valued function τ: $\gamma_1 \rightarrow R$ by letting $\tau(x)$ = smallest positive number τ such that $\eta_\tau(x) \in S$. Clearly τ is well defined, since for $t \neq t'$ and $\eta_t(x)$, $\eta_{t'}(x) \in S$ one must have $|t - t'| > 2\varepsilon$. The preceding remark shows that, as x varies in γ_1, $\eta_{\tau(x)}(x)$ keeps away from ∂S. This implies that τ is a continuous function on γ_1. Indeed, a simple application of the implicit function theorem will show that τ is differentiable.

The map $x \longrightarrow \eta_{\tau(x)}(x)$ defines a diffeomorphism of γ_1 onto a simple closed curve $\gamma_2 \subset S$. Unlike γ_1, γ_2 may or may not be an orbit of X. In any case γ_2 cuts each generatrix of S transversally hence exactly once, so γ_2 is homotopic to γ_1 in S. Consequently, if we show that $\gamma_1 \cap \gamma_2 = \varnothing$, there will be a ring A in S, bounded by γ_1 and γ_2. This is proved next.

We have assumed, by contradiction, that the φ-orbit containing γ_1 is not a torus, so it is a continuous one to one image of a cylinder. Under these circumstances, γ_1 and γ_2 must be disjoint, because the existence of a common point $y \in \gamma_1 \cap \gamma_2$ would give $y = \eta_\tau(x)$ for some $x \in \gamma_1$ and $\tau = \tau(x)$. On the other hand, $y = \xi_s(x)$, $s \in R$. Hence

$$x = \xi_{-s}\eta_\tau(x) = \varphi_r(x), \quad r = (-s, \tau) .$$

Since $\tau \neq 0$, the vectors $(s_1, 0)$ and $(-s, \tau)$ are linearly independent. They both belong to the isotropy group of x under the action φ, so this group is free on 2 generators, and the φ-orbit containing γ_1 is a torus, contrary to the assumption.

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Notice that we are assuming actually that no φ-orbit in μ is a torus, so if $x \in \mu$ and $t \neq t'$, one must have $\eta_t(x) \neq \xi_s\eta_{t'}(x)$ no matter which $s \in R$ is chosen.

Consider now the compact cylinder $B = \{\eta_t(x); x \in \gamma_1, 0 \leqq t \leqq \tau(x)\}$. Together with the ring $A \subset S$, bounded by γ_1 and γ_2, B forms a topological torus $T = A \cup B$, embedded in the manifold M. Now we apply Lemma 1. Since M is simply-connected, $H_2(M; Z_2) = 0$, by Poincaré duality. Hence T bounds in M. Lemma 1 then provides us with the following: $M - T = C_1 \cup C_2$ is the disjoint union of 2 connected components, each of which has T as its point-set boundary. Moreover, if $\varepsilon > 0$ is the number introduced in the definition of S, and $\mathring{A} = A - B$ (the absolute interior of A), then the set $\{\eta_t(x); x \in \mathring{A}, 0 < t \leqq \varepsilon\}$ lies entirely in one component, say C_1, whereas the set $\{\eta_t(x); x \in \mathring{A}, \quad \varepsilon \leqq t < 0\}$ is contained in C_2.

We need, however, a little more than this, namely: for any $y \in \gamma_1$, and $-\varepsilon \leqq t < 0$, $\eta_t(y) \in C_2$. (Similarly, if $z \in \gamma_2$ and $0 < t \leqq \varepsilon$, then $\eta_t(z) \in C_1$.) To prove this, let $u \longrightarrow x(u)$, $0 \leqq u \leqq 1$, be a path in A, going from a point $x(0) \in \mathring{A}$ to the given y, without touching any other point of B besides y. Then $u \longrightarrow \eta_t(x(u))$ is a path in M, starting at $\eta_t(x(0)) \in C_1$ and never touching T for $u < 1$. So, $\eta_t(y) = \eta_t(x(1))$ either belongs to T (that is, to B) or to C_1. But it is clear that $\eta_t(y) \notin B$, so $\eta_t(y) \in C_2$.

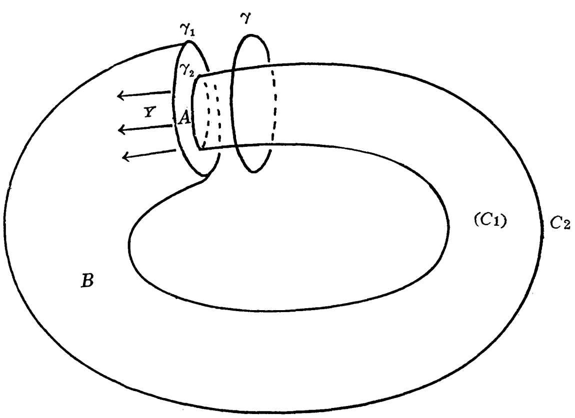

Fig. 2

Let $\gamma = \eta_{-\varepsilon}(\gamma_1)$, so γ is a closed orbit of X, entirely contained in C_2. Choose a point $w \in \gamma$. As we have seen earlier, there are arbitrarily large positive values of t for which $\eta_t(w)$ returns to any pre-assigned neighborhood of γ. In particular, $\eta_t(w)$ must return to C_2 for some values of t strictly greater than $\tau(w) + \varepsilon$. Now, for $\tau(w) + \varepsilon < t < \tau(w) + 2\varepsilon$, $\eta_t(w) \in C_1$. Therefore, $\eta_t(w)$ may only leave C_1 and enter C_2 for values $t > \tau(w) + 2\varepsilon$. But how? First of all, $\eta_t(w)$ cannot cross $\mathring{A}$ from C_1 to C_2 because all the stream lines of the flow

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η in $\mathring{A}$ lead to C_1. Secondly, $\eta_t(w)$ cannot touch B for values $t > \tau(w) + 2\varepsilon$ because this would give $\eta_t(w) = \xi_s\eta_{t'}(w)$ with $t' \leqq \tau(x) + \varepsilon$, hence $t \neq t'$. This, however, contradicts a previous observation and it is the final contradiction that proves the theorem.

3. Fields of 2-frames in 3-manifolds

Let $V_2(R^3)$ denote the Stiefel manifold of orthonormal 2-frames in euclidean space R^3. The map $\{e_1, e_2\} \to \{e_1, e_2, e_1 \times e_2\}$, where $e_1 \times e_2$ is the vector product, defines a diffeomorphism of $V_2(R^3)$ onto the set of all positively oriented orthonormal bases of R^3, that is, onto the group SO(3) of all real orthogonal 3×3 matrices with determinant $+1$. Therefore, the fundamental group of $V_2(R^3)$ has precisely 2 elements and its non-zero homotopy class is represented by the closed path $t \to \{e_1(t), e_2(t)\}$, where $e_1(t) = (-\sin 2\pi t, \cos 2\pi t, 0)$, $e_2(t) = (0, 0, 1)$, $0 \leqq t \leqq 1$. (See [4, p. 149].) Now consider $V_2'(R^3)$, the manifold of *all* 2-frames in R^3. It is geometrically clear that $V_2(R^3)$ is a deformation retract of $V_2'(R^3)$, so these manifolds have the same fundamental group, and its generator is still represented by the same path as before.

(A) Let $X \subset R^3$. A continuous *field of 2-frames* on X is a continuous map $f\colon X \to V_2'(R^3)$. Denote by D^2 the unit disc of the plane $z = 0$ in R^3 and by $S^1 = \partial D^2$ its boundary circle. The continuous map $f\colon S^1 \to V_2'(R^3)$, given by $f(x, y, 0) = \{e_1, e_2\}$, with $e_1 = (-y, x, 0)$, $e_2 = (0, 0, 1)$, when considered as a closed path, agrees with the generator of $\pi_1(V_2'(R^3))$ above described, so it is not homotopic to a constant map, that is, it cannot be extended to a continuous map $\bar{f}\colon D^2 \to V_2'(R^3)$. In other words, *there is no continuous field of 2-frames on D^2 that agrees with f on S^1.*

We indicate next some consequences of this elementary remark. The first one is a simplification of Reinhart's argument in [6, pp. 186–187].

(B) Consider the solid torus $N^3 = N$, in R^3 generated by rotating the disc $D^2 = \{(x, y, 0) \in R^3;\ x^2 + y^2 \leqq 1\}$ around, say, the axis $y = 2$, $z = 0$. Let $T^2 = T = \partial N$ denote the 2-torus generated by the rotation of $S^1 = \partial D^2$. The various positions occupied by S^1 during the rotation are the *meridian* circles of T and the circles described by each given point of S^1 are called *parallels*. The meridians are homotopic to zero in N, the parallels represent a generator of $\pi_1(N) = Z$. In T itself, parallels cannot be told apart from meridians and, together, they represent the two free generators of $\pi_1(T) = Z + Z$. Let X and Y denote orthonormal unit vector fields on T such that X is tangent to the meridians and Y is tangent to the parallels of T. *There is no continuous field of 2-frames on N that agrees with $\{X, Y\}$ on T.* In fact, there is not even a field of 2-frames on D^2 that agrees with X and Y on S^1.

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In order to endow the sphere S^3 with a foliated structure of class C^∞ and dimension 2, G. Reeb [5] has introduced a foliation of the solid 3-torus N whose leaves are $T = \partial N$ and planar leaves filling $N - T$. It is not difficult to define a continuous field of 2-frames $\{X, Y\}$ on N, such that X and Y, at each point of N, are tangent to the Reeb foliation. It is not possible, however, to choose X and Y with the additional property that $[X, Y] \equiv 0$.

More generally, the following is true:

(C) Let N be a compact 3-manifold whose boundary is a 2-dimensional torus $T = \partial N$, such that the homomorphism $h: \pi_1(T) \to \pi_1(N)$, induced by the inclusion $T \subset N$ is *not* a monomorphism. *Then there exists no pair of commuting vector fields X, Y on N which are linearly independent at each point and tangent to the boundary T.*

The proof uses the *loop theorem* of C. Papakyriakopoulos [8], according to which there exists a simple closed curve $C \subset T$ which is not homotopic to a constant in T, but is the boundary of a topological disc $D \subset N$. We may assume that C and D are differentiably embedded. (This follows, for instance, from [2, Th. (6.3), p. 544].) We may also assume that D is transversal to T, hence $D \cap T = C$, since we could have started with the boundary of a tubular neighborhood of T in N, instead of T. Now assume that there exist linearly independent commuting vector fields X, Y on N which are tangent to T. They define a differentiable action φ of R^2 on N such that each orbit has dimension 2, hence the boundary torus T is an orbit of φ. Therefore, if we fix at random a point $x_0 \in T$, the isotropy group of x_0 is a discrete (free abelian) subgroup G_0 of R^2, with 2 generators. The map of R^2 onto T, defined by $r \to \varphi_r(x_0)$ induces, by passage to the quotient, a diffeomorphism $R^2/G_0 \approx T$. This diffeomorphism sends the quotient image of any line parallel to a non-zero vector $(a, b) \in R^2$ onto an orbit of the vector field $U = aX + bY$ in T. Since we may represent any element of the fundamental group of R^2/G_0 by the image of a straight line in R^2, it follows then that by choosing a and b conveniently in $U = aX + bY$, we can make all the U-orbits of points in T be homotopic to the given simple closed curve C. Once chosen a, b, there is no difficulty in choosing c, d, so that the vector field $V = cX + dY$ is linearly independent of U. Clearly there is a diffeomorphism of T which is isotopic to the identity and carries, say, the U-orbit of x_0 onto C. Such diffeomorphism may then be extended to a diffeomorphism of N, hence the U-orbit of x_0 is, just as C, the boundary of a differentiable 2-disc E contained in N. A tubular neighborhood of E in N is diffeomorphic to the set

$$P = \{(x, y, z) \in R^3; x^2 + y^2 \leqq 1, \ -1 < z < +1\} .$$

This diffeomorphism carries E onto D^2 and lets the vector fields U, V define a

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field of 2-frames on P which, when restricted to D^2, gives the one that was precluded by the initial remark (A).

4. The main theorem

In the proof of Theorem 3, an important tool is the following result:

THEOREM 2 (Haefliger). *A differentiable foliation of codimension one on a compact, simply-connected manifold has at least one leaf that is not simply-connected.*

PROOF. This follows readily from [1, Prop. p. 390]. Indeed, every differentiable foliation of codimension one on a compact manifold admits a closed transversal: just consider a solution curve C of the field of line elements orthogonal to the given foliation. By compactness, the ω-limit set of the solution curve C is non-empty, so it contains a point x. Take a neighborhood V of x, trivially fibered by arcs of the orthogonal trajectories to the foliation. The curve C will cut the neighborhood V along countable many intervals. One may easily connect two consecutive intervals of $C \cap V$ so as to obtain a differentiable simple closed curve γ, transversal to the leaves of the foliation. Since the manifold is simply-connected γ is homotopic to a constant. By the mentioned Proposition in [1], there exists a leaf F of the foliation and a loop in F such that the germ of homeomorphism of R corresponding to that loop *via* holonomy is not the identity. Since the holonomy is invariant under homotopy (see [1, p. 378]), such loop is not homotopic to a constant in F, hence the leaf F is not simply-connected.

The main theorem may now be proved.

THEOREM 3. *Every compact differentiable* 3-*manifold with finite fundamental group has rank one.*

PROOF. Since every odd-dimensional manifold has rank at least one, the above statement means that if M is a compact differentiable 3-manifold with finite fundamental group, any 2 commuting vector fields X, Y on M must be linearly dependent at some point. The universal covering space of M is a compact simply-connected 3-manifold to which the given vector fields may be lifted, so we do not lose any generality by assuming, as we do from now on, that M is simply-connected.

Let then X, Y be commuting differentiable vector fields on M. Suppose (as if it were possible) that X, Y are linearly independent at each point of M. For each $x \in M$, let D_x denote the vector subspace of M_x spanned by X and Y. The map $D\colon x \to D_x$ defines a 2-dimensional sub-bundle of the tangent bundle. This tangent sub-bundle is integrable because of the relation $[X, Y] \equiv 0$

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(cf. S. Lang. Introduction to Differentiable Manifolds, Ch. VI). The maximal integral submanifolds of D are 2-dimensional leaves of a foliation of M. By Theorem 2, this foliation has a non-simply-connected leaf. Now the leaves of our foliation are also the orbits of the action $\varphi: R^2 \times M \rightarrow M$ induced by the commuting vector fields X, Y. Therefore, a non-simply-connected leaf must be a (one to one continuous image of) cylinder or a torus. Let $x_0 \in M$ be a point whose φ-orbit is a cyclinder or a torus. The isotropy group G_0 of x_0 is discrete and not zero. Let $(a, b) \in R^2$ be a free generator of G_0. Replacing X, Y by $X' = aX + bY$, $Y' = cX + dY$, so that the constants a, b, c, d satisfy $ad - bc \neq 0$, we have X', Y' as linearly independent commuting vector fields on M, such that the X'-orbit of x_0 is closed. Change notation, going back to X, Y, instead of X', Y'.

By Theorem 1, there is a φ-orbit in M which is diffeomorphic to a 2-torus T. Let M_1 and M_2 be the two 3-manifolds with boundary T such that $M = M_1 \cup M_2$ and $M_1 \cap M_2 = T$. At least one of the inclusion homomorphisms $\pi_1(T) \rightarrow \pi_1(M_1)$, $\pi_1(T) \rightarrow \pi_1(M_2)$ has a non-zero kernel for otherwise, by Van Kampen's theorem [7, p. 177], $\pi_1(M)$ would be the free product of $\pi_1(M_1)$ and $\pi_1(M_2)$, with amalgamation of the subgroup (isomorphic to) $\pi_1(T)$. Then $\pi_1(M)$ would contain a subgroup isomorphic to $\pi_1(T)$ (*Schreier's theorem*) which is contradictory, since $\pi_1(M) = 0$. Let N be one of the manifolds M_1, M_2 such that the injection $\pi_1(T) \rightarrow \pi_1(N)$ has a non-zero kernel. We are in the situation of § 3, paragraph (C), hence the commuting vector fields X, Y must be linearly dependent somewhere in N. This concludes the proof of the theorem.

REMARK. It may be that any simply-connected compact 3-manifold is homeomorphic with the sphere S^3. If this is the case (or if one wishes to prove the above theorem only for manifolds that are covered by S^3), then, instead of the Loop Theorem, ony may use the theorem of Alexander (Proc. Nat. Acad. Sci. (1924), pp. 6–8) according to which any 2-torus T piecewise linearly embedded in S^3 is such that at least one of the components of $S^3 - T$ is a solid torus.

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