COMMUTING VECTOR FIELDS ON S^2

ELON L. LIMA[1]

1. **Introduction.** The aim of this note is to prove Theorem B below. (See §2 for definitions.) The proof makes essential use of the Poincaré-Bendixon theorem [1, p. 391]. A continuous version of this classical theorem is also true. It follows, for instance, from Theorem 11 in [2]. This enables us to prove Theorem A below, which implies Theorem B.

THEOREM A. *Every continuous action of the additive group R^2 on the sphere S^2 admits a fixed point.*

COROLLARY. *The same fact is true for the projective plane P^2, instead of S^2.*

THEOREM B. *Let X, Y be vector fields of class C^k, $k \geqq 1$, on the sphere S^2, with $[X, Y] \equiv 0$. There exists a point $p \in S^2$ such that $X(p) = Y(p) = 0$.*

COROLLARY. *The above theorem holds also for the projective plane P^2.*

The same type of argument used to prove Theorem A, together with a simple induction procedure, will show that every continuous action of the additive group R^m ($m \geqq 1$) on S^2 (and hence on P^2) has a fixed point. Thus, any finite set $X_1, \cdots, X_m$ of pairwise commuting vector fields on S^2 (or P^2) has a common singularity.

Of course, there are differentiable actions of R^m without fixed points on the torus T^2 and on the Klein bottle K^2, for every $m \geqq 1$. The natural action of R^2 on T^2 has only one orbit, which has dimension 2. Every continuous nontrivial action of R^2 on K^2 must have at least one 1-dimensional orbit. There is one differentiable action of R^2 on K^2 with exactly two orbits or dimensions 2 and 1, respectively.

It seems plausible that every continuous action of R^2 on a compact 2-manifold, other than T^2 or K^2, must have a fixed point, but, this question is open at present.

Added in proof. Meanwhile, I have been able to prove this conjecture. See the Research Announcement in the Bull. Amer. Math. Soc. of May 1963, pp. 366–368.

I wish to thank Robert Ellis for interesting conversations.

Received by the editors November 12, 1962.

[1] Fellow of the Guggenheim Foundation. Work done under partial support of the National Science Foundation.

2. **Definitions and proof that A implies B.** Let T be an additive group and M a set. An *action* of T on M is a map $\phi: T\times M\to M$ such that, writing $\phi(t, p)=\phi_t(p)$, one has $\phi_{s+t}(p)=\phi_s(\phi_t(p))$ and $\phi_0(p)=p$, for all $s, t\in T$, $p\in M$, 0 = neutral element of T. Then, for each $t\in T$, the map $\phi_t: M\to M$, given by $p\to\phi_t(p)=\phi(t, p)$, is a 1-1 correspondence whose inverse is ϕ_{-t}. When T is a topological group and M is a topological space, we talk about a *continuous action* ϕ, meaning that $\phi: T\times M\to M$ is continuous. Then, each $\phi_t: M\to M$ is a homeomorphism. In case T is a Lie group and M is a differentiable manifold, there is the notion of a differentiable action; each ϕ_t is then a diffeomorphism.

For example, let X be a vector field of class C^k, $k\geqq 1$, on a compact differentiable manifold M. Integrating X, one obtains a map $\xi: R\times M\to M$, where $\xi_s(p)$, for every real s and every $p\in M$, is the point with parameter value s on the trajectory of X that starts at p. The map ξ defines a differentiable action of the additive group R on M [**3**, pp. 65, 66]. Conversely, given a differentiable action ξ of R on M, one obtains a vector field X on M, defined by $X(p)$ = tangent vector, at $t=0$, of the parametrized curve $t\to\xi_t(p)$.

Let now Y be another vector field, of class C^k, $k\geqq 1$, on M. Call η the action of R on M determined by Y. The *Lie bracket* $[X, Y]$ is a vector field on M, for which the following well-known lemma holds.

LEMMA. $[X, Y]\equiv 0$ *on* M *if, and only if,* $\xi_s\circ\eta_t=\eta_t\circ\xi_s$ *for all* $s, t\in R$.

Based on this fact, we say that two vector fields X, Y on M *commute* when their Lie bracket $[X, Y]$ vanishes identically on M.

A pair of commuting vector fields on a compact differentiable manifold M yields a differentiable action of the additive group R^2 on M. In fact, the two vector fields generate actions $\xi, \eta: R\times M\to M$, and we define the action $\phi: R^2\times M\to M$ by $\phi_r(p)=\xi_s(\eta_t(p))=\eta_t(\xi_s(p))$, for $p\in M$, $r=(s, t)\in R^2$. Conversely, given a differentiable action $\phi: R^2\times M\to M$, one obtains two commuting vector fields X, Y on M as follows. Let R' and R'', respectively, denote the x- and the y-axis of R^2. Let $\xi=\phi|(R'\times M)$ and $\eta=\phi|(R''\times M)$ be the restrictions of the action ϕ. Then ξ and η are actions of the reals on M, which define, as above, the vector fields X, Y. These fields commute, by the preceding lemma.

The *orbit* of a point $p\in M$ under a continuous action $\phi: T\times M\to M$ is the set $T(p)=\{\phi_t(p);\ t\in T\}$. The point p will be called a *fixed point* of the action ϕ when $T(p)=\{p\}$. Let ϕ be the action of R^2 generated by two differentiable vector fields on a compact differentiable manifold M. Then $p\in M$ is a fixed point of ϕ if, and only if, $X(p)$

$=Y(p)=0$. This follows directly from the uniqueness theorem for ordinary differential equations.

We see, therefore, that Theorem A implies Theorem B.

A continuous action of R^2 on M induces an action of T on M, for every subgroup $T\subset R^2$. Let R' be the x-axis and R'' be the y-axis in R^2. Then a point $p\in M$ is fixed under R^2 if, and only if, it is fixed simultaneously under the induced actions of R' and R'' on M. More generally, if p is fixed under the action of R'', then its orbit $R'(p)$ consists entirely of fixed points of R''. Of course, all limit points of the orbit $R'(p)$ will then be fixed under R'' too.

3. **Proof of Theorem A.** It is known that every continuous action of the reals on S^2 has a fixed point. Let then $p\in S^2$ be a fixed point under the y-axis R'', which acts on S^2 as explained above. Consider the orbit $R'(p)$. It consists entirely of fixed points under R'', and so does the set of limit points of this orbit. If there is a fixed point q of R' in the closure of $R'(p)$, q will be fixed under R', R'' and hence under R^2, proving the theorem. If not, by Poincaré-Bendixon, the ω-limit set of the orbit $R'(p)$ is a simple closed curve C', in fact a closed orbit of R', whose points are all fixed under R''. The curve C' bounds a disc D' in S^2. Observe that D' is invariant under R^2, since C' is also an orbit under R^2. We shall now use transfinite induction to show that there exists a fixed point of R^2 inside D'.

Consider the collection $\mathfrak{D}$ of all closed discs D in S^2, such that the boundary $C=\partial D$ is a closed orbit of R', whose points are all fixed under R''. The first part of the proof shows that $\mathfrak{D}$ is not empty. Let $\mathfrak{D}$ be partially ordered by inclusion. Notice that for $D_\lambda\neq D_\mu$ in $\mathfrak{D}$, $D_\lambda\subset D_\mu$ is the same as $D_\lambda\subset \operatorname{int}(D_\mu)$. By the Hausdorff maximal principle, let $\{D_\lambda\}$ be a maximal chain in $\mathfrak{D}$, and write $C_\lambda=\partial D_\lambda$. Choose $x\in\bigcap D_\lambda$ such that $x=\lim x_n$, $x_n\in C_{\lambda(n)}$. (In order to do this, take a net $\{y_\lambda\}$ with $y_\lambda\in C_\lambda$ and let $x_n=y_{\lambda(n)}$ be a convergent subnet. Let $x=\lim x_n$. Since $\{\lambda(n)\}$ is cofinal in $\{\lambda\}$, we have $x\in\bigcap D_\lambda$.) Then x is a fixed point of R'', and so is every point of the closure of its orbit $\mathrm{Cl}(R'(x))$. Notice that $\mathrm{Cl}(R'(x))\subset D_\lambda$ for each λ. We may assume that $\mathrm{Cl}(R'(x))$ contains no fixed points of R'. Then, the limit set of $R'(x)$ is $C_0\cup C_1$, each C_i being a closed orbit of R', whose points are all fixed under R''. Both C_0, C_1 are contained in all discs D_λ. Now, either C_0 is disjoint of C_1, or $C_0=C_1=R'(x)$. Moreover, given λ, either $C_i=C_\lambda$ or $C_i\subset\operatorname{int}(D_\lambda)$. Suppose first $C_0\cap C_1=\emptyset$. Then, it is easily seen that at least one of them, say C_0, is contained in the interior of all discs D_λ. So C_0 bounds a disc D_0, which belongs to $\mathfrak{D}$ and is properly contained in all D_λ, thus contradicting the maximality of the chain $\{D_\lambda\}$.

Assume next the other case: $C_0 = C_1$. If $C_0 = C_1 \subset \operatorname{int}(D_\lambda)$ for all λ, we come to a contradiction exactly as above. Otherwise, $C_0 = C_1 = C_{\lambda_0}$ for some λ_0, and then $D_0 = D_{\lambda_0}$ is the smallest disc in the chain. The group R' acts on D_0, leaving its boundary C_0 invariant, and has no fixed point on C_0. Then R' has a fixed point y inside D_0. In the ω-limit set of the orbit $R''(y)$ we find a closed orbit $\overline{C}$ of R'', pointwise fixed under R', and, finally, inside $\overline{C}$, by the same argument, there is a closed orbit of R', pointwise fixed under R'', which will provide a contradiction against the maximality of the chain $\{D_\lambda\}$. This completes the proof of Theorem A.

COROLLARY. *Every continuous action of R^2 on the projective plane P^2 has a fixed point.*

PROOF. In fact, it is a simple exercise in covering space theory to see that every action of a simply connected topological group G on a space X can be covered uniquely by a continuous action of G on any covering space of X. So, given an action of R^2 on P^2, it can be covered by an action of R^2 on S^2, which has a fixed point. The projection of this fixed point will be fixed under the original action on P^2.

BIBLIOGRAPHY

1. E. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

2. W. Kaplan, *Regular curve-families filling the plane*. I, Duke Math. J. **7** (1940), 154–185.

3. S. Lang, *Introduction to differentiable manifolds*, Interscience, New York, 1962.

INSTITUTE FOR ADVANCED STUDY AND
INSTITUTO DE MATEMÁTICA PURA E APLICADA, RIO DE JANEIRO, BRAZIL